Logical Construction of Final Coalgebras*

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Abstract

We prove that every finitary polynomial endofunctor of a category $\mathcal{C}$ has a final coalgebra if $\mathcal{C}$ is locally Cartesian closed, has finite disjoint coproducts and a natural number object. More generally, we prove that the category of coalgebras for such an endofunctor has all finite limits.

Introduction

A goal of this paper is to prove that every polynomial endofunctor

$$P(X) = \sum_{i=1,\ldots,n} \Omega_i \times X^{A_i} \tag{1}$$

of a given category $\mathcal{C}$ has a final coalgebra, assuming that (1) $\mathcal{C}$ is locally Cartesian closed, (2) it has finite disjoint coproducts, (3) it has a natural number object. This statement follows from a more general result which also shows that the category of P-coalgebras has all finite limits. An immediate consequence of this statement is that the functor P is completely iterative in the sense of [2] and generates a cofree comonad. To this end observe that the collection of polynomial endofunctors is closed w.r.t. addition and multiplication by a fixed object A of $\mathcal{C}$ (the latter up to natural isomorphism), and recall that a comonad cofreely generated by P is essentially the same as the parametrized final coalgebra of $A \times P(X)$.

Our proof is inspired by a particular set theoretic representation – see [17, 10] for example – of the final P-coalgebra which we illustrate next.

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Example 0.1. Consider the functor $P(X) = \Omega_1 \times X + \Omega_2 \times X^2$, where $\Omega_1 = \{f\}$ and $\Omega_2 = \{g\}$. Its final coalgebra is the set of finite and infinite terms over the signature $\{f, g\} = \Omega_1 + \Omega_1$, f a unary function symbol and g a binary function symbol. Let now $A = \{(f, 1), (g, 1), (g, 2)\}$ be the disjoint sum of the arities of the signature and let $\Omega = \{f, g, \bot\}$ be obtained from the previous signature by addition of the new symbol $\bot$. The infinite terms over the signature $\{f, g\}$ are in bijection with those complete A-branching trees labeled in Ω satisfying the following clauses:

1. If a node is labeled by f, then the son in the direction $(f, 1)$ is not labeled by the symbol $\bot$ while all the sons in the directions $(g, 1), (g, 2)$ are labeled by $\bot$.

2. If a node is labeled by g, then the son in the direction $(f, 1)$ is labeled by the symbol $\bot$ and the sons in the directions $(g, 1), (g, 2)$ are not labeled by $\bot$.

3. The root of the tree is not labeled by the symbol $\bot$.

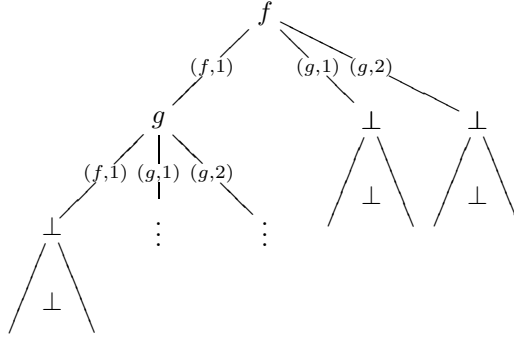A complete tree satisfying these conditions is represented in figure 1.

Figure 1: Infinite terms as complete trees

This example can obviously be generalized to arbitrary set-theoretic polynomial endofunctors and arbitrary signatures. If we let $A = \sum_i A_i$ and $\Omega = \{\bot\} + \sum_i \Omega_i$, the set-theoretic final P-coalgebra is a subset of the function space Ω^{A^*}. This function space can also be seen as the set of complete A-branching trees with labels in Ω; as such it carries the structure of a final coalgebra for the functor $Q(X) = \Omega \times X^A$.

We want to investigate the process by which a final P-coalgebra is extracted from a final Q-coalgebra by means of logical operations. The outcome of our investigation can be synthesized as follows. A full and faithful 'completion' functor K from the category of P-coalgebras to the category of Q-coalgebras is defined. This functor factors as a full and faithful right adjoint K^+ followed

by a full and faithful left adjoint I. This suffices to argue that the category of P-coalgebras has finite limits provided that the category of Q-coalgebras has finite limits – a standard lemma in the theory of reflective and coreflective subcategories. Also, under our assumptions, the category of Q-coalgebras has indeed all finite limits.

This extraction process can be carried out by means of the weak internal logic corresponding to the categorical properties (1)-(3). This is the minimal logic in which the clauses of Example 0.1 are expressible. For example, the use there of classical logic is restricted to the constructive Boolean logic of extensive categories, see [15]. Interestingly, clauses (1)-(2) point to the existence of the left adjoint I while clause (3) of 0.1 is related to the right adjoint K^+.

Our interest in this problem is part of a general investigation [16, 29] on the relationships between induction and coinduction, i.e. initial algebras and final coalgebras of functors. In [16] we proved that, given an adjoint pair of endofunctors $F \dashv G$, if a free F-algebra functor $\widehat{F}$ exists and has a right adjoint $\widetilde{G}$, then $\widetilde{G}$ is a cofree G-coalgebra functor.[1] For example, if A^* is the free monoid generated by A in a Cartesian closed category, then $A^* \times X$ is the free $A \times (-)$-algebra generated by X, and the function space X^{A^*} turns out to be the cofree $(-)^A$-coalgebra over X. Therefore if a locally Cartesian closed category has a natural number object N, then it has the free monoids A^* – they are computed as partial products[2] of A with the arrow $\mathsf{N} \times \mathsf{N} \xrightarrow{+} \mathsf{N} \xrightarrow{s} \mathsf{N}$ – and consequently it has final $(-)^A$-coalgebras. Briefly, some final coalgebras arise from duality.

A question suggested from the set theoretic example but left open in [16] was whether and under which conditions the duality is the starting point for constructing other final coalgebras. The following result, expressed within the framework of [16], gives a positive answer to this question: given adjoint pairs $F_i \dashv G_i$, $i = 1, \ldots, n$, the category of coalgebras for the endofunctor

$$P(X) = \sum_{i=1,\ldots,n} \Omega_i \times G_i(X)$$

has all finite limits under the assumption that $\mathcal{C}$ has the properties (1)-(2) and that both a free algebra functor of $\sum_i F_i$ and a cofree coalgebra functor of $\prod_i G_i$ exist. We recover the previous statement letting $F_i X = A_i \times X$, $G_i X = X^{A_i}$. Using this framework it is possible to argue that functorial systems of equations such as

$$\left\{ X_k = \sum_{i=1,\ldots,n} \Omega_{i,k} \times X_{f(k)}^{A_{i,k}} \right\}_{k=1,\ldots,m}$$

have a greatest solution. To this end observe that conditions (1)-(2) are stable under formation of products of categories and use results from [22].

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[1] The converse holds too: if both a free F-algebra functor $\widehat{F}$ and a cofree G-coalgebra functor $\widetilde{G}$ exist, then $\widehat{F}$ is left adjoint to $\widetilde{G}$, see [6, 5, 16].

[2] See [19].

A motivation for this work has been that of deriving existence of final coalgebras on axiomatic bases, without either imposing strong completeness properties on our models, nor relying on the full topos theoretic categorical structure. As a matter of fact the construction presented here depends only on a restricted fragment of this structure: we do not need a subobject classifier nor a factorization system corresponding to existential quantification.

In concrete categories – mainly locally presentable categories [4] – final coalgebras are usually constructed as limits of the sequence of iterated functorial applications beginning at the terminal object [8, 3, 34]. For these categories final coalgebras of polynomial functors can be constructed in this way. On the other hand, several categories studied in computer science lack the completeness properties required to successfully perform the terminal sequence construction. A priori, we list among them the effective topos [20] and the free topos generated by a countable language [26]. Yet, elementary toposes admit final coalgebras of partial product functors, thus of polynomial functors: in [23] their final coalgebras are constructed as internal limits of the terminal sequence. Existence of these internal limits depends on the development of internal category theory and on the theory of iterative data in an elementary topos [25].

The full topos theoretic structure is often considered too strong, in particular when designing programming languages with a categorical semantics in mind – see [12, 13] and the language Charity for an example – or considering categorical universes of predicative mathematics [27]. Locally Cartesian closed categories with disjoint coproducts have often been proposed as an alternative choice [22, 1]. The reason for not assuming completeness stems from considerations on initial semantics of typed programming languages. These are usually interpreted in categories with some kind of structure, and, among these structured categories, the initial one plays the role of a canonical model. This model will inevitably lack the completeness properties: for example, a natural number object in such a category cannot be a countable coproduct of the terminal object, since the homsets are at most countable.[3] We finally mention that recursive-theoretic categories such as ω-$Sets$ and PER are locally Cartesian closed without being complete or topoi [21, §1.2.7].

The paper is structured as follows: in section 1 we overview the mathematical setting and introduce the notation. In section 2 we represent the categories of P-coalgebras and Q-coalgebras in an 'automatic' form, exploiting extensiveness of $\mathcal{C}$. This allows to easily define the completion functor K, its factorization, and to argue that its first factor K^+ has a left adjoint. In section 3 we make use of locally Cartesian closeness to internally characterize the domain of the second factor I. In the sequent section 4 we show that the factor I has a right adjoint. We add concluding remarks in section 5.

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[3]This remark is due to R. Cockett.

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1   Preliminaries

The goal of this paper is to prove the following statement:

Theorem 1.1. *Let $\mathcal{C}$ be a locally Cartesian closed category with finite disjoint coproducts. Let $F_i \dashv G_i$, $i \in I$, be a finite collection of adjoint endofunctors of $\mathcal{C}$ and define*

$$F(X) = \sum_{i \in I} F_i(X)\,, \qquad\qquad G(X) = \prod_{i \in I} G_i(X)\,.$$

If a pair of adjoint endofunctors $\widehat{F} \dashv \widetilde{G}$ – where $\widehat{F}$ is a free F-algebra functor or equivalently $\widetilde{G}$ is a cofree G-coalgebra functor – exists, then the functor

$$P(X) = \sum_{i \in I} \Omega_i \times G_i(X) \tag{2}$$

has a final coalgebra. More generally, the category of P-coalgebras has all finite limits.

By examining the proof the reader will convince himself that if $\mathcal{C}$ has enough completeness properties, then the statement holds for an indexing set I of a given infinite cardinality. We begin explaining the statement and recalling some basic results we will need later.

A category $\mathcal{C}$ is *locally Cartesian closed* if it has a terminal object 1 and each slice category $\mathcal{C}/C$ is a Cartesian closed category [18, 28, 31]. Hence $\mathcal{C}$ is itself Cartesian closed, being equivalent to the slice $\mathcal{C}/1$. If restricted to monic arrows with same codomain C, the local Cartesian closed structure endows the collection of all subobjects of C with the structure of a Brouwerian semilattice: it has finite intersections $\bigwedge$ and an implication operation $\rightarrow$ satisfying usual axioms. Moreover pulling back along an arrow preserves this structure.

A locally Cartesian closed $\mathcal{C}$ is a *distributive* category, meaning that if we construct the pullbacks

where the $\mathtt{in}_i : B_i \longrightarrow \sum_i B_i$ are coproduct injections, then the diagram $(\pi_i : P_i \longrightarrow A)_{i \in I}$ is again a coproduct. $\mathcal{C}$ is an *extensive* category if the converse condition holds: if $(\pi_i : P_i \longrightarrow A)_{i \in I}$ is a coproduct diagram and the diagram above commutes, then it is a pullback. Rephrased, coproduct injections are Cartesian natural transformations. A locally Cartesian closed category has finite disjoint coproducts (coproducts are disjoint if the intersection of distinct

coproduct injections is an initial object) if and only it has finite coproducts and is an extensive category.[4] The following property of distributive categories, see [14], will be frequently used:

Lemma 1.2. *In a distrubutive category coproduct injections are monic.*

An F-*algebra* is a pair (Q, s) with $s : FQ \longrightarrow Q$. A morphism of F-algebras from (Q, s) to (Q', s') is an arrow $f : Q \longrightarrow Q'$ such that $s \cdot f = Ff \cdot s'$. F-algebras and their morphisms form a category $\mathrm{Alg}(F)$. The category $\mathrm{CoAlg}(G)$ of G-*coalgebras* is defined dually. By saying that a *free F-algebra functor* exists we mean that the forgetful functor $U_F : \mathrm{Alg}(F) \longrightarrow \mathcal{C}$, sending (Q, s) to Q, has a left adjoint. Spelled out, for every object X we can find an object $\widehat{F}X$ and a diagram

$$\mathtt{z}_X : X \longrightarrow \widehat{F}X \qquad\qquad \mathtt{s}_X : F\widehat{F}X \longrightarrow \widehat{F}X$$

with the initial property w.r.t. similar diagrams: for every pair (a, f), where $a : X \longrightarrow A$ and $f : FA \longrightarrow A$, there exists a unique arrow $\{\!|a, f|\!\} : \widehat{F}X \longrightarrow A$ such that

$$\mathtt{z}_X \cdot \{\!|a, f|\!\} = a\,, \qquad\qquad \mathtt{s}_X \cdot \{\!|a, f|\!\} = F\{\!|a, f|\!\} \cdot f\,.$$

Similarly, by saying that a cofree G-coalgebra functor exists, we mean that the forgetful functor $U_G : \mathrm{CoAlg}(G) \longrightarrow \mathcal{C}$ has a right adjoint. Spelled out, for every object X we can find an object $\widetilde{G}X$ and a diagram

$$\mathtt{h}_X : \widetilde{G}X \longrightarrow X \qquad\qquad \mathtt{t}_X : \widetilde{G}X \longrightarrow G\widetilde{G}X$$

with the final property w.r.t. similar diagrams. Clearly, $\widehat{F}$ and $\widetilde{G}$ are functors, obtained by composing the left adjoint with U_F and the right adjoint with U_G; $\mathtt{z}, \mathtt{s}, \mathtt{h}, \mathtt{t}$ are natural transformations.

If a free F-algebra functor $\widehat{F}$ is given and we define

$$\mathtt{i}_X = F\mathtt{z}_X \cdot \mathtt{s}_X : FX \longrightarrow \widehat{F}X\,, \quad \mathtt{m}_X = \{\!|\mathtt{id}_{\widehat{F}X}, \mathtt{s}_X|\!\} : \widehat{F}\widehat{F}X \longrightarrow \widehat{F}X\,,$$

then the tuple $\langle \widehat{F}, \mathtt{i}, \mathtt{z}, \mathtt{m} \rangle$ is the *free monad* generated by F [7, 33]. Dually, if a cofree G-coalgebra functor $\widetilde{G}$ is given, then we can define $\mathtt{p}_X : \widetilde{G}X \longrightarrow GX$ and $\mathtt{d}_X : \widetilde{G}X \longrightarrow \widetilde{G}\widetilde{G}X$ so that $\langle \widetilde{G}, \mathtt{p}, \mathtt{h}, \mathtt{d} \rangle$ is the cofree comonad generated by G. Let $F \dashv G$ be a pair of adjoint endofunctors of an arbitrary category. In [16] we proved the following facts. If $\widehat{F}$ is a free F-algebra functor and $\widetilde{G}$ is a cofree G-coalgebra functor, then $\widehat{F}$ is left adjoint to $\widetilde{G}$ – see also [6, 5]. Conversely, if a free F-algebra functor $\widehat{F}$ is given and has a right adjoint $\widetilde{G}$, then $\widetilde{G}$ can be endowed with the structure of a cofree G-coalgebra functor. Dually: if a cofree G-coalgebra functor $\widetilde{G}$ is given and has a left adjoint $\widehat{F}$, then $\widehat{F}$ can be can be endowed with the structure of a free F-algebra functor. The proof of these fact relied on the following well known isomorphisms, see [6, 7] and [9, §3.7]:

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[4]In the following we will only use the fact that coproduct injections are Cartesian natural transformation.

Lemma 1.3. *The category* $\mathrm{Alg}(F)$ *of algebras for an endofunctor F is isomorphic to:*

1. *the category* $\mathrm{CoAlg}(G)$ *for a functor G right adjoint to F;*

2. *the Eilenberg-Moore category of algebras for $(\widehat{F}, \mathbf{z}, \mathbf{m})$, the free monad generated by F;*

3. *if $\widetilde{G}$ is right adjoint to $\widehat{F}$, the Eilenberg-Moore category of coalgebras for the comonad on $\widetilde{G}$ dual to $(\widehat{F}, \mathbf{z}, \mathbf{m})$.*

These isomorphisms commute with the respective forgetful functors.

We recall the nature of these isomorphisms. The first isomorphism sends an algebra $s : FX \longrightarrow X$ to its transpose $s^\flat : X \longrightarrow GX$.[5] The second isomorphism is obtained by sending an algebra s to $\widehat{s} = \{\!\!|\mathrm{id}_X, s|\!\!\}$, the unique arrow $\widehat{s} : \widehat{F}X \longrightarrow X$ such that $\mathbf{z}_X \cdot \widehat{s} = \mathrm{id}_X$ and $\mathbf{s}_X \cdot \widehat{s} = F\widehat{s} \cdot s$; it is easily seen that this is an algebra for the monad on $\widehat{F}$. Finally, if $(\widehat{F}, \mathbf{z}, \mathbf{m})$ is a monad and $\widetilde{G}$ is right adjoint to $\widehat{F}$ with $\mathbf{ev}$ the counit of the adjunction, the dual comonad on $\widetilde{G}$ is defined as follows: $\mathbf{h}_X = \mathbf{z}_{\widetilde{G}X} \cdot \mathbf{ev}_X$ is the counit of the cofree comonad while the comultiplication $\mathbf{d}_X : \widetilde{G}X \longrightarrow \widetilde{G}\widetilde{G}X$ is obtained by transposing twice the arrow

$$\mathbf{m}_{\widetilde{G}X} \cdot \mathbf{ev}_X : \widehat{F}\widehat{F}\widetilde{G}X \longrightarrow \widehat{F}\widetilde{G}X \longrightarrow X \,.$$

The isomorphism between algebras and coalgebras of 1.3.i restricts to an isomorphism between Eilenberg-Moore algebras and Eilenberg-Moore coalgebras.

We shall use the following fact: given an endofunctor F of a category $\mathcal{C}$, we can define a new functor from the slice category $\mathcal{C}/C$ to the slice category $\mathcal{C}/FC$: an object (X, x) – where $x : X \longrightarrow C$ – is sent to (FX, Fx) and an arrow $f : X \longrightarrow Y$ such that $f \cdot y = x$ is sent to Ff. The following lemma is well known [24, §4.1.3].

Lemma 1.4. *If F has a right adjoint G and $\mathcal{C}$ has pullbacks then the functor $F : \mathcal{C}/C \longrightarrow \mathcal{C}/FC$ has also a right adjoint $\forall_F$ which is computed by pulling back along the unit of the adjunction:*

$$
\begin{array}{ccc}
\forall_F Q & \longrightarrow & GQ \\
\Big\downarrow{\scriptstyle \forall_F q} & & \Big\downarrow{\scriptstyle Gq} \\
C & \underset{\eta_C}{\longrightarrow} & GFC
\end{array}
$$

Observe that if q is monic then Gq is also monic, and therefore $\forall_F q$ is monic.

We conclude this section introducing the notation. In the statement of Theorem 1.1 the functor F is the sum $\sum_{i \in I} F_i$; hence we shall consider several

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[5] We shall use in this paper the notation $(\cdot)^\flat$ for the transpose of arrows under the adjunction, with $(\cdot)^\#$ for the inverse correspondence.

restrictions of the multiplication $\mathtt{m}$ and of the action $s : FX \longrightarrow X$ of an arbitrary F-algebra:

$$\mathtt{m}_i = \mathtt{in}_{F_i\widehat{F}} \cdot \mathtt{i}_{\widehat{F}} \cdot \mathtt{m} \qquad\qquad s_i = \mathtt{in}_{F_iX} \cdot \mathtt{i}_X \cdot \widehat{s}$$
$$= \mathtt{in}_{F_i\widehat{F}} \cdot \mathtt{s}_{\widehat{F}} : F_i\widehat{F} \longrightarrow \widehat{F}, \qquad = \mathtt{in}_{F_iX} \cdot s_X : F_iX \longrightarrow X \,.$$

We shall occasionally say that X is a subobject of Y, in which case we shall use ι_Y^X for the intended monic arrow $X \longrightarrow Y$; we shall only write ι^X if Y is understood.

2   Coalgebras as Automata, a Reflector

The category $\mathcal{C}$ being extensive, we can represent coalgebras of polynomial endofunctors by means of tuples, in an analogous way we usually represent coalgebras as automata in the category of sets and functions. We make explicit this correspondence in this section: we define a category $\mathrm{Aut}(P)$ of P-automata and argue it is equivalent to the category of P-coalgebras, P being the functor defined in (2). Similarly we define a category $\mathrm{Aut}(Q)$ of Q-automata for the functor

$$Q(X) = \Omega \times \prod_{i \in I} G_i(X) \,,$$

where Ω is $1 + \sum_{i \in I} \Omega_i$, and we argue it is equivalent to the category of Q-coalgebras. We define then a 'completion' functor

$$K : \mathrm{Aut}(P) \longrightarrow \mathrm{Aut}(Q)$$

and begin its study. We shall eventually see that this functor is responsible for the existence of a terminal object and finite limits in $\mathrm{Aut}(P)$ – and therefore in $\mathrm{CoAlg}(P)$.

We introduce here the following notation: we let $J = \{0\} \cup I$, with $0 \notin I$, $\Omega_0 = 1$ is a terminal object, and for $K \subseteq J$ we let $\Omega_K = \sum_{k \in K} \Omega_k$, so that $\Omega = \Omega_J = 1 + \Omega_I$. A similar notation is used for arbitrary families $\{Q_j\}_{j \in J}$, for example Q_I stands for $\sum_{i \in I} Q_i$.

Definition 2.1. The category $\mathrm{Aut}(P)$ of P-*automata* is defined as follows:

- An object of $\mathrm{Aut}(P)$ is a tuple

$$\mathcal{A} = \langle\, \{\mathtt{in}_i : Q_i \longrightarrow Q\}_{i \in I}, \{s_i\}_{i \in I}, \{h_i\}_{i \in I} \,\rangle$$

  where:

  – $\{\mathtt{in}_i : Q_i \longrightarrow Q\}_{i \in I}$ is a coproduct diagram in $\mathcal{C}$,

  – $s_i : F_iQ_i \longrightarrow Q$ for each $i \in I$,

  – $h_i : A_i \longrightarrow \Omega_i$ for each $i \in I$.

- An arrow f from $\mathcal{A}$ to $\mathcal{B}$ is an arrow $f : Q^{\mathcal{A}} \longrightarrow Q^{\mathcal{B}}$ in $\mathcal{C}$ such that:

    - $\mathtt{in}_i^{\mathcal{A}} \cdot f$ factors (uniquely because of Lemma 1.2) through $\mathtt{in}_i^{\mathcal{B}}$:

    $$\mathtt{in}_i^{\mathcal{A}} \cdot f = f_i \cdot \mathtt{in}_i^{\mathcal{B}} \, ,$$

    - for each $i \in I$ the following equations hold:

    $$s_i^{\mathcal{A}} \cdot f = F_i f_i \cdot s_i^{\mathcal{B}} \, , \qquad\qquad h_i^{\mathcal{A}} = f_i \cdot h_i^{\mathcal{B}} \, .$$

We shall occasionally use the standard terminology for automata: Q is the carrier of $\mathcal{A}$, the s_i are actions, and the h_i are labeling arrows.

Definition 2.2. The category $\mathrm{Aut}(Q)$ of Q-*automata* is defined as follows:

- An object of $\mathrm{Aut}(Q)$ is a tuple

    $$\mathcal{A} = \big\langle\, \{\mathtt{in}_j : Q_j \longrightarrow Q\}_{j \in J}, \{s_{i,j}\}_{i \in I, j \in J}, \{h_i\}_{i \in I} \,\big\rangle$$

    where:

    - $\{\mathtt{in}_j : Q_j \longrightarrow Q\}_{j \in J}$ is a coproduct diagram in $\mathcal{C}$,
    - $s_{i,j} : F_i Q_j \longrightarrow Q$ for each $i \in I$ and $j \in J$,
    - $h_i : Q_i \longrightarrow \Omega_i$ for each $i \in I$.

- An arrow $f : \mathcal{A} \longrightarrow \mathcal{B}$ is an arrow $f : Q^{\mathcal{A}} \longrightarrow Q^{\mathcal{B}}$ in $\mathcal{C}$ such that:

    - for each $j \in J$, $\mathtt{in}_j^{\mathcal{A}} \cdot f$ factors (necessarily uniquely) through $\mathtt{in}_j^{\mathcal{B}}$:

    $$\mathtt{in}_j^{\mathcal{A}} \cdot f = f_j \cdot \mathtt{in}_j^{\mathcal{B}} \, ,$$

    - for each $i \in I$ and $j \in J$ the following equations hold:

    $$s_{i,j}^{\mathcal{A}} \cdot f = F_i f_j \cdot s_{i,j}^{\mathcal{B}} \, , \qquad\qquad h_i^{\mathcal{A}} = f_i \cdot h_i^{\mathcal{B}} \, .$$

Lemma 2.3. *The categories* $\mathrm{Aut}(P)$ *and* $\mathrm{CoAlg}(P)$ *are equivalent.*

Proof. Given a P-coalgebra $\beta : B \longrightarrow \sum_{i \in I} \Omega_i \times G_i B$, we construct a co-product diagram $\mathtt{in}_i : B_i \longrightarrow B$ and arrows $\langle h_i, t_i \rangle : B_i \longrightarrow \Omega_i \times G_i B$ by pulling back along injections $\mathtt{in}_i : \Omega_i \times G_i B \longrightarrow \sum_{i \in I} \Omega_i \times G_i B$; we obtain a P-automaton $\mathcal{B}$ by transposing the t_i. This is the object part of a full and faithful functor $R : \mathrm{CoAlg}(P) \longrightarrow \mathrm{Aut}(P)$ that is the identity on morphisms. In the other direction, given a P-automaton $\mathcal{A}$ we define the coalgebra

$$\sum_{i \in I} \langle h_i, s_i^{\flat} \rangle : Q^{\mathcal{A}} \cong \sum_{i \in I} Q_i^{\mathcal{A}} \longrightarrow \sum_{i \in I} \Omega_i \times G_i Q^{\mathcal{A}} \, .$$

An arrow f in $\mathrm{Aut}(P)$ is also a coalgebra morphism, thus this construction defines a functor $L : \mathrm{Aut}(P) \longrightarrow \mathrm{CoAlg}(P)$ which is left adjoint to R.

$\mathrm{CoAlg}(P)$ is therefore a reflective subcategory of $\mathrm{Aut}(P)$, and this depends only on distributivity of the base category $\mathcal{C}$. In order to conclude that the two categories are equivalent it is must be argued that each object of $\mathrm{Aut}(P)$ is isomorphic to an object coming from $\mathrm{CoAlg}(P)$; this is a consequence of coproduct injections being Cartesian natural transformations in an extensive category. $\qquad\square$

In view of Lemma 1.3, it should be easy to argue that the category of Q-coalgebras is isomorphic to the category of F-algebras equipped with an arrow from the carrier to Ω, and to the category of Eilenberg-Moore algebras for the monad $\widehat{F}$ equipped again with an arrow from the carrier to Ω. Using the first isomorphism we are going to argue that:

Lemma 2.4. *The category Q-coalgebras is equivalent to the category of Q-automata.*

Proof. Given an F-coalgebra $s : FB \longrightarrow B$ and an arrow $h : B \longrightarrow \Omega$, we construct a coproduct diagram $\mathrm{in}_j : B_j \longrightarrow B$ and arrows $h_j : B_j \longrightarrow \Omega_j$ by pulling back along the coproduct diagram $\mathrm{in}_j : \Omega_j \longrightarrow \Omega$; we obtain a Q-automaton $\mathcal{B}$ by letting $s_{i,j} = \mathrm{in}_{F_i B_j} \cdot F\mathrm{in}_j \cdot s$. Again, this is the object part of a full and faithful functor that is the identity on morphisms. Its left adjoint is described as follows: given a Q-automaton $\mathcal{A}$ we define $h =\ !\ + \sum_{i \in I} h_i : Q^{\mathcal{A}} \longrightarrow \Omega$. Recall that $(F_i \mathrm{in}_j : F_i Q_j^{\mathcal{A}} \longrightarrow F_i Q^{\mathcal{A}})_{j \in J}$ is a coproduct diagram, since F_i is a left adjoint. Therefore we define $s_i : F_i Q^{\mathcal{A}} \longrightarrow Q^{\mathcal{A}}$ by saying that $F_i \mathrm{in}_j \cdot s_i = s_{i,j}$ and $s : FQ^{\mathcal{A}} \longrightarrow Q^{\mathcal{A}}$ by $\mathrm{in}_i \cdot s = s_i$.

The category of F-algebras equipped with an arrow to Ω is therefore a reflective subcategory of $\mathrm{Aut}(Q)$, assuming that $\mathcal{C}$ is only distributive. The two categories are equivalent if $\mathcal{C}$ is extensive as well. $\qquad\square$

We are ready to define the functor K. In automata theoretic terms, it amounts to completing a partial deterministic automaton by adding a new sink state.

Definition 2.5. For a P-automaton $\mathcal{A}$ the Q-automaton $K(\mathcal{A})$ is defined as follows. We let $Q_0^{K(\mathcal{A})}$ be the terminal object, and $Q_i^{K(\mathcal{A})} = Q_i^{\mathcal{A}}$ for $i \in I$; the coproduct diagram is given by:

$$1 \xrightarrow{\mathrm{in}_l} 1 + Q^{\mathcal{A}} \xleftarrow{\mathrm{in}_r} Q^{\mathcal{A}} \xleftarrow{\mathrm{in}_i^{\mathcal{A}}} Q_i^{\mathcal{A}} \ .$$

The actions $s_{i,j}$ are defined as follows:

$$s_{i,j} = \begin{cases} s_i^{\mathcal{A}} \cdot \mathrm{in}_r \colon F_i Q_i^{\mathcal{A}} \longrightarrow Q^{\mathcal{A}} \longrightarrow 1 + Q^{\mathcal{A}} \,, & i = j \,, \\ !\cdot \mathrm{in}_l \colon F_i Q_j^{\mathcal{A}} \longrightarrow 1 \longrightarrow 1 + Q^{\mathcal{A}} \,, & \text{otherwise.} \end{cases}$$

Finally, the labeling is unchanged: $h_i^{K(\mathcal{A})} = h_i^{\mathcal{A}}$. Clearly, if f is a morphism in $\mathrm{Aut}(P)$, then $1 + f$ is a morphism in $\mathrm{Aut}(Q)$, so that the construction K defines a functor from $\mathrm{Aut}(P)$ to $\mathrm{Aut}(Q)$.

Lemma 2.6. *The functor K is full and faithful.*

Proof. The functor is faithful since f is determined from $1 + f$ by pulling back along the right coproduct injection. On the other hand, if $g : K(\mathcal{A}) \longrightarrow K(\mathcal{B})$ is a morphism, then $\mathtt{in}_i^{\mathcal{A}} \cdot \mathtt{in}_r \cdot g = g_i \cdot \mathtt{in}_i^{\mathcal{B}} \cdot \mathtt{in}_r$, and $\mathtt{in}_l \cdot g = g_0 \cdot \mathtt{in}_l$ where g_0 is necessarily the identity of the terminal object. Therefore we can write $g = 1 + g'$, and g' is a morphism in $\mathrm{Aut}(Q)$. For example

$$s_i^{\mathcal{A}} \cdot g' \cdot \mathtt{in}_r = s_i^{\mathcal{A}} \cdot \mathtt{in}_r \cdot g = s_{i,i}^{K(\mathcal{A})} \cdot g = F_i g_i \cdot s_{i,i}^{K(\mathcal{B})} = F_i g_i \cdot s_i^{\mathcal{B}} \cdot \mathtt{in}_r$$

implies $s_i^{\mathcal{A}} \cdot g' = F_i g_i \cdot s_i^{\mathcal{B}}$, since coproduct injections are monic. $\square$

Definition 2.7. We denote by $\delta\text{-}\mathrm{Aut}(Q)$ the full subcategory category of $\mathrm{Aut}(Q)$ of those Q-automata $\mathcal{B}$ for which $s_{i,i}^{\mathcal{B}}$ factors through $Q_I^{\mathcal{B}}$ and $s_{i,j}^{\mathcal{B}}$ factors through $Q_0^{\mathcal{B}}$ for $i \neq j$.

The functor K lands in $\delta\text{-}\mathrm{Aut}(Q)$, thus it can be factored as

$$\mathrm{Aut}(P) \xrightarrow{K^+} \delta\text{-}\mathrm{Aut}(Q) \xrightarrow{I} \mathrm{Aut}(Q),$$

where the last functor I is the inclusion. Since K and I are both full and faithful, it follows that K^+ is full and faithful too. Thus the image of the functor K in the category $\mathrm{Aut}(Q)$ can be characterized as follows: an object $\mathcal{B}$ of $\mathrm{Aut}(Q)$ is isomorphic to an object of the form $K(\mathcal{A})$ if and only if

- it lies in the full subcategory $\delta\text{-}\mathrm{Aut}(Q)$,

- $Q_0^{\mathcal{B}}$ is a terminal object in $\mathcal{C}$.

Proposition 2.8. *The functor K^+ has a left adjoint L.*

The left adjoint $L(\mathcal{A})$ is obtained by restricting all the structure from J to I: the underlying coproduct is now $\mathtt{in}_i : Q_i \longrightarrow Q_I$; by the definition of $\delta\text{-}\mathrm{Aut}(Q)$ we can write $s_{i,i} = s_i' \cdot \mathtt{in}_I$ (and such a factorization is unique) so that the actions are defined to be these s_i', while the labeling arrows are unchanged. Again, using the fact that coproduct injections are monic, it is easily seen that this construction is functorial and that, for a Q-automaton $\mathcal{A}$ in $\delta\text{-}\mathrm{Aut}(Q)$ and a P-automaton $\mathcal{B}$, $f : Q_I^{\mathcal{A}} \longrightarrow Q^{\mathcal{B}}$ is a morphism in $\mathrm{Aut}(P)$ if and only if $! + f : Q^{\mathcal{A}} \longrightarrow 1 + Q^{\mathcal{B}}$ is a morphism in $\mathrm{Aut}(Q)$.

Corollary 2.9. *If the category $\delta\text{-}\mathrm{Aut}(Q)$ has finite limits, then the category $\mathrm{Aut}(P)$ has also finite limits.*

The previous proposition has shown that we can identify $\mathrm{Aut}(P)$ with a reflective subcategory of $\delta\text{-}\mathrm{Aut}(Q)$. It is therefore enough to recall that a replete reflective subcategory is closed under existing limits [11, §3.5.3].

We shall argue in the next sections that $\delta\text{-}\mathrm{Aut}(Q)$ has indeed all finite limits, so that the proposition holds without the proviso.

3 A Characterization of δ-$\mathrm{Aut}(Q)$

In this section we propose an alternative characterization of Q-automata in δ-$\mathrm{Aut}(Q)$. To this goal, we need a preliminary observation: for an arbitrary object $\mathcal{A}$ of $\mathrm{Aut}(Q)$ we consider the two pullback squares:

$$
\begin{array}{ccccc}
Q_j & \xrightarrow{\ \psi_j\ } & P_j & \xrightarrow{\ \pi_j\ } & Q_j \\
\downarrow{\scriptstyle \mathtt{in}_j} & {\scriptstyle (3)} & \downarrow{\scriptstyle \pi_{\widehat{F}Q}} & {\scriptstyle (4)} & \downarrow{\scriptstyle \mathtt{in}_j} \\
Q & \xrightarrow{\ \mathtt{z}_Q\ } & \widehat{F}Q & \xrightarrow{\ \widehat{s}\ } & Q
\end{array} \quad .
$$

The square (4) is a pullback by definition. The square (3) is a pullback for the following reason. Because of the unit law of Eilenberg-Moore algebras, the relation $\mathtt{z}_Q \cdot \widehat{s} = \mathtt{id}_Q$ holds. Also the commutative diagram corresponding to the relation $\mathtt{in}_j \cdot \mathtt{id}_Q = \mathtt{id}_{Q_j} \cdot \mathtt{in}_j$ is a pullback since $\mathtt{in}_j$ is monic. Hence (3) is a pullback and moreover $\psi_j \cdot \pi_j = \mathtt{id}_{Q_j}$. As a consequence the diagram

$$
\begin{array}{ccc}
F_i P_j & \xrightarrow{\ F_i \pi_j\ } & F_i Q_j \\
\downarrow{\scriptstyle F_i \pi_{\widehat{F}Q}} & & \downarrow{\scriptstyle s_{i,j}} \\
F_i \widehat{F}Q & \xrightarrow{\ \mathtt{m}_i \cdot \widehat{s}\ } & Q
\end{array}
$$

commutes, which can be seen as follows:

$$
\begin{aligned}
F_i \pi_{\widehat{F}Q} \cdot \mathtt{m}_i \cdot \widehat{s} = F_i \pi_{\widehat{F}Q} \cdot F_i \widehat{s} \cdot s_i &= F_i(\pi_{\widehat{F}Q} \cdot \widehat{s}) \cdot s_i \\
&= F_i(\pi_j \cdot \mathtt{in}_j) \cdot s_i & \text{by (4)} \\
&= F_i \pi_j \cdot s_{i,j} . & \text{by Lemma 2.4}
\end{aligned}
$$

Considering that for an object in δ-$\mathrm{Aut}(Q)$ we have $s_{i,i} = s'_{i,i} \cdot \mathtt{in}_I$ and $s_{i,j} = s'_{i,j} \cdot \mathtt{in}_0$ if $i \neq j$, we see that the diagrams

$$
\begin{array}{ccccc}
F_i P_i & \xrightarrow{F_i \pi_i} & F_i Q_i & \xrightarrow{s'_{i,i}} & Q_I \\
\big\uparrow{\scriptstyle F_i \pi_{\widehat{F}Q}} & & & & \downarrow{\scriptstyle \mathtt{in}_I} \\
F_i \widehat{F}Q & & \xrightarrow{\ \mathtt{m}_i \cdot \widehat{s}\ } & & Q
\end{array}
\qquad
\begin{array}{ccccc}
F_i P_j & \xrightarrow{F_i \pi_j} & F_i Q_j & \xrightarrow{s'_{i,j}} & Q_0 \\
\big\uparrow{\scriptstyle F_i \pi_{\widehat{F}Q}} & & & & \downarrow{\scriptstyle \mathtt{in}_0} \\
F_i \widehat{F}Q & & \xrightarrow{\ \mathtt{m}_i \cdot \widehat{s}\ } & & Q
\end{array}
\quad i \neq j
$$

commute. If we define

$$
P_{i,j} = \text{pullback of } \mathtt{m}_i \cdot \widehat{s} \text{ against } \begin{cases} \mathtt{in}_I & \text{if } i = j , \\ \mathtt{in}_0 & \text{otherwise} , \end{cases}
$$

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we obtain that the arrow $\widehat{F}_i \pi_{\widehat{F}Q}$ factors through the subobject $P_{i,j}$ of $F_i \widehat{F} Q$. Transposing this relation according to Lemma 1.4, we deduce that, for each $j \in J$ and $i \in I$, the relation

$$P_j \leq \forall_{F_i} P_{i,j} \tag{5}$$

holds in the Brouwerian semilattice of subobjects of $\widehat{F}Q$. Conversely, suppose that $P_j \leq \forall_{F_i} P_{i,j}$, that is we can write $F_i \pi_j \cdot s_{i,j} = F_i \pi_{\widehat{F}Q} \cdot \mathtt{m}_i \cdot \widehat{s} = \alpha \cdot \mathtt{in}_d$, where $d = I$ if $i = j$ and $d = 0$ otherwise. It follows that $s_{i,j} = F_i \psi_j \cdot F_i \pi_j \cdot s_{i,j} = F_i \psi_j \cdot \alpha \cdot \mathtt{in}_d$, which shows that $s_{i,j}$ factors through the proper coproduct injection. Thus we have shown that:

Proposition 3.1. *An object $\mathcal{A}$ of $\mathrm{Aut}(Q)$ belongs to $\delta\text{-}\mathrm{Aut}(Q)$ if and only if, for each $i \in I$ and $j \in J$, the relation* (5) *holds in the Brouwerian semilattice of subobjects of $\widehat{F}Q^{\mathcal{A}}$.*

4 The Coreflector

The observations of the previous section suggest the following construction to be performed on an arbitrary Q-automaton $\mathcal{A}$. Its building blocks are subobjects P_i of $\widehat{F}Q^{\mathcal{A}}$ and $P_{i,j}$ of $F_i \widehat{F} Q^{\mathcal{A}}$ defined as the following pullbacks:

The P_j and the $P_{i,j}$ are indeed subobjects of $\widehat{F}Q$ and $F_i \widehat{F}Q$, respectively, because of Lemma 1.2. In the Brouwerian semilattice of subobjetcs of $Q^{\mathcal{A}}$, we define:

$$C_j = \forall_{\widehat{F}} (P_j \to \bigwedge_{i \in I} \forall_{F_i} P_{i,j}), \qquad\qquad D = \bigwedge_{j \in J} C_j .$$

The meaning of the universal quantification $\forall_F$ and the reason for which C_j is a subobject of $Q^{\mathcal{A}}$ are explained in Lemma 1.4.

In order to understand the definition of the C_j, we consider the set theoretic Example 0.1. Here $Q = \Omega^{A^*}$ is the set of all A-complete trees labeled either

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by the symbol $\perp$ or by some element in some Ω_i; $\widehat{s} : A^* \times \Omega^{A^*} \longrightarrow \Omega^{A^*}$ is correspondence which takes the pair (w, t) to the subtree of t rooted at w, the function $\lambda x.t(wx)$, and $h : \Omega^{A^*} \longrightarrow \Omega$ is evaluation at the empty word. $P_{i,i}$ is seen to be the set of triples (a, w, t) in $A_i \times A^* \times \Omega^{A^*}$ such that $t(wa)$ belongs to Ω_i for some $i \in I$, and $P_{i,j}$ – for $i \neq j$ – is the set of triples (a, w, t) in $A_j \times A^* \times \Omega^{A^*}$ such that the subtree of t rooted at wa is labeled by $\perp$. Therefore C_j is the collection of trees $t \in \Omega^{A^*}$ with the following property: *for all $w \in A^*$ such that $t(w) \in \Omega_j$, for all $i \in I$ and $a \in A_i$, if $i = j$ then $t(wa)$ is in Ω_I, and otherwise, if $i \neq j$, then $t(wa) = \perp$.* That is, the definition of C_j mimics clauses (1)-(2) of 0.1.

Lemma 4.1. *The object D is a subautomaton of $\mathcal{A}$.*

Proof. We shall show that the arrow $\widehat{F}\iota^{C_j} \cdot \widehat{s}$ factors through C_j; this will be enough since it is easily verified that the intersection of subobjects closed under the action of $\widehat{F}$ is again closed under this action. By the definition of C_j, this is equivalent to the pullback of P_j along $\widehat{F}(\widehat{F}\iota^{C_j} \cdot \widehat{s})$ to factor through $\forall_{F_i} P_{i,j}$, for all $i \in I$. To this end, in the diagram below the arrow ψ corresponds under several adjunctions to the identity of C_j. This implies that we can factor the pullback P through $\forall_{F_i} P_{i,j}$:
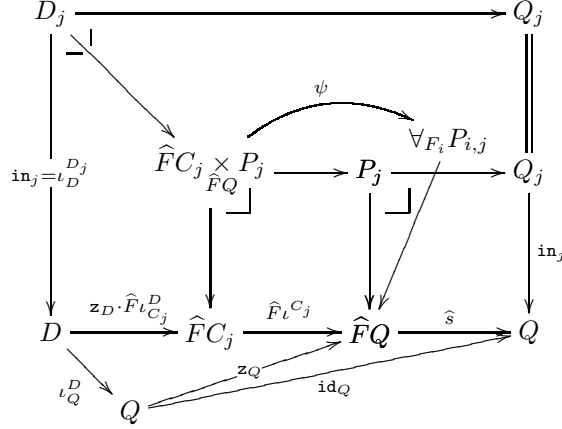

The statement of the lemma follows since the relations

$$\mathbf{m}_{C_j} \cdot \widehat{F}\iota^{C_j} \cdot \widehat{s} = \widehat{F}\widehat{F}\iota^{C_j} \cdot \mathbf{m}_Q \cdot \widehat{s} = \widehat{F}\widehat{F}\iota^{C_j} \cdot \widehat{F}\widehat{s} \cdot \widehat{s}$$

exhibit P as the pullback of P_j along $\widehat{F}(\widehat{F}\iota^{C_j} \cdot \widehat{s})$. $\square$

Proposition 4.2. *The automaton D belongs to the category δ-Aut(Q).*

Proof. The diagram

shows that the arrow $\iota_Q^{D_j} \cdot \mathbf{z}_Q$ can be factored through $\forall_{F_i} P_{i,j}$ over $\widehat{F}Q$. Transposing this relation, we obtain that $F_i(\iota_Q^{D_j} \cdot \mathbf{z}_Q)$ can be factored through $P_{i,j}$ over $F_i\widehat{F}Q$. Considering the definition of the $P_{i,j}$ as pullbacks, we obtain that $F_i(\iota_Q^{D_j} \cdot \mathbf{z}_Q) \cdot \mathbf{m}_i \cdot \widehat{s}$ can be factored through $\mathbf{in}_I : Q_I \longrightarrow Q$ if $i = j$ and through $\mathbf{in}_0 : Q_0 \longrightarrow Q$ otherwise. Observe that $F_i \mathbf{z}_Q \cdot \mathbf{m}_i \cdot \widehat{s} = s_i$ and that $F_i(\iota_Q^{D_j} \cdot \mathbf{z}_Q) \cdot \mathbf{m}_i \cdot \widehat{s} = F_i \iota_Q^{D_j} \cdot s_i = s_{i,j}^{D(\mathcal{A})} \cdot \iota_Q^D$, thus: $s_{i,j}^{D(\mathcal{A})} \cdot \iota_Q^D$ can be factored through $\mathbf{in}_I : Q_I \longrightarrow Q$ if $i = j$ and through $\mathbf{in}_0 : Q_0 \longrightarrow Q$ otherwise. Since D_0 and D_I are obtained by pulling back Q_0 and Q_I against ι_Q^D, we obtain the statement of the proposition. □

Proposition 4.3. *Let $\mathcal{A}, \mathcal{B}$ be two automata, of which $\mathcal{A}$ is in δ-$\mathrm{Aut}(Q)$. If $f : \mathcal{A} \longrightarrow \mathcal{B}$ is a morphism, then it factors through $D(\mathcal{B})$.*

Proof. It is an easy exercise to show that if $\iota : \mathcal{C} \longrightarrow \mathcal{B}$ is monic, and the morphism $f : \mathcal{A} \longrightarrow \mathcal{B}$ factors through $Q^{\mathcal{C}}$ in the underlying category $\mathcal{C}$, $f = f' \cdot \iota$, then f' is also a morphism from $\mathcal{A}$ to $\mathcal{C}$ in $\mathrm{Aut}(Q)$.

Therefore we shall prove that f factors through $D = Q^{D(\mathcal{B})}$, by showing that it factors through each $C_j^{\mathcal{B}}$. Unraveling its definition, we need to show that $\widehat{F}f$ factors through $P_j^{\mathcal{B}} \to \bigwedge_{i \in I} \forall_{F_i} P_{i,j}^{\mathcal{B}}$, or equivalently that the pullback P_j of $\widehat{F}f \cdot \widehat{s}^{\mathcal{B}}$ along the injection $Q_j^{\mathcal{B}} \longrightarrow Q^{\mathcal{B}}$ factors through each $\forall_{F_i} P_{i,j}^{\mathcal{B}}$ over $\widehat{F}Q^{\mathcal{B}}$. Since $\widehat{F}f \cdot \widehat{s}^{\mathcal{B}} = \widehat{s}^{\mathcal{A}} \cdot f$ and $h^{\mathcal{A}} = f \cdot h^{\mathcal{B}}$, this pullback is $P_j^{\mathcal{A}}$. Since $\mathcal{A}$ belongs to δ-$\mathrm{Aut}(Q)$, we know from Proposition 3.1 that $P_j^{\mathcal{A}}$ factors through $\forall_{F_i} P_{i,j}^{\mathcal{A}}$ over $\widehat{F}Q^{\mathcal{A}}$. In order to reach our goal, we only need to argue that the construction $\forall_{F_i} P_{i,j}^{\mathcal{X}}$ is natural in $\mathcal{X}$. Clearly the $P_{i,j}^{\mathcal{X}}$ are natural in $\mathcal{X}$, and transposing the

diagram

$$\begin{array}{ccc}
F_i \forall_{F_i} P_{i,j}^{\mathcal{A}} & \longrightarrow & P_{i,j}^{\mathcal{A}} & \longrightarrow & P_{i,j}^{\mathcal{B}} \\
& \searrow & \downarrow & & \downarrow \\
& & F_i \widehat{F} Q^{\mathcal{A}} & \xrightarrow{F_i \widehat{F} f} & F_i \widehat{F} Q^{\mathcal{B}}
\end{array}$$

whose triangle on the left is the counit of the adjunction, we derive the factorization we are looking for. □

To conclude this section, we resume what we have proved:

Theorem 4.4. *The category δ-$\mathrm{Aut}(Q)$ is a coreflective subcategory of $\mathrm{Aut}(Q)$, the coreflector being the construction D.*

Corollary 4.5. *If $\mathrm{Aut}(Q)$ has finite limits, then δ-$\mathrm{Aut}(Q)$ has finite limits.*

It is a standard fact [11, §3.5.3] that the limit of a diagram $\{\mathcal{A}_k\}_{k \in K}$ in δ-$\mathrm{Aut}(Q)$ can be calculated by applying the coreflection to the Q-automaton $\lim_k \mathcal{A}_k$.

We can now argue that $\mathrm{Aut}(Q)$ has all finite limits, so that the statements of Corollaries 2.9 and 4.5 hold without their proviso.

Proposition 4.6. *The category $\mathrm{Aut}(Q)$ has all finite limits.*

Proof. We have argued that $\mathrm{Aut}(Q)$ is isomorphic to the category of Eilenberg-Moore algebras for the free monad on F equipped with an arrow to Ω. Since we are assuming that an F-algebra, call it $\widetilde{G}\Omega$, cofree over Ω exists, we end-up establishing the following isomorphism of categories:

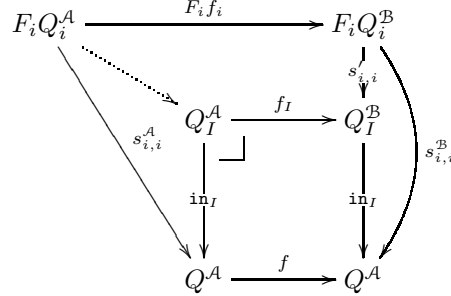$$\mathrm{Aut}(Q) \cong \mathcal{C}^{\mathsf{T}}/\widetilde{G}\Omega\,,$$

where $\mathsf{T} = (\widehat{F}, \mathbf{z}, \mathbf{m})$ is the free monad on F. That is, we have identified $\mathrm{Aut}(Q)$ with a certain slice category. Existence of finite limits follows then from existence of finite limits in $\mathcal{C}^{\mathsf{T}}$ which are created from the forgetful functor. □

5 Further Observations

5.1 Behavioral Characterization of δ-$\mathrm{Aut}(Q)$.

The category δ-$\mathrm{Aut}(Q)$ has the following property: if $\mathcal{B}$ is in δ-$\mathrm{Aut}(Q)$ and $f : \mathcal{A} \longrightarrow \mathcal{B}$ is an arrow of $\mathrm{Aut}(Q)$, then $\mathcal{A}$ is also in δ-$\mathrm{Aut}(Q)$. This can be seen by considering the following diagram showing that if $s_{i,i}^{\mathcal{B}}$ factors through

$\mathtt{in}_I$, then so does $s_{i,i}^{\mathcal{A}}$ (a similar diagram is used for $s_{i,j}$ when $i \neq j$):

$$
\begin{array}{ccc}
F_i Q_i^{\mathcal{A}} & \xrightarrow{\ F_i f_i\ } & F_i Q_i^{\mathcal{B}} \\
 & & \Big\downarrow{s_{i,i}} \\
 & Q_I^{\mathcal{A}} \xrightarrow{\ f_I\ } Q_I^{\mathcal{B}} & \\
 & \mathtt{in}_I \Big\downarrow \quad \Big\downarrow \mathtt{in}_I & s_{i,i}^{\mathcal{B}} \\
 & Q^{\mathcal{A}} \xrightarrow{\ f\ } Q^{\mathcal{A}} &
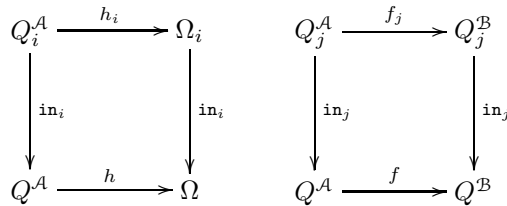\end{array}
$$

Hence we can identify $\delta\text{-Aut}(Q)$ with the slice category $\text{Aut}(Q)/D(\widetilde{G}\Omega)$:

Proposition 5.1. *A Q-automaton belongs to δ-Aut(Q) if and only if the unique arrow to the terminal Q-automaton $\widetilde{G}\Omega$ factors through $D(\widetilde{G}\Omega)$.*

In particular, the coreflection D can be described as pulling back along the monomorphism $\iota_{\widetilde{G}\Omega}^{D(\widetilde{G}\Omega)} : D(\widetilde{G}\Omega) \longrightarrow \widetilde{G}\Omega$. If we think of the unique arrow from a Q-automaton to the terminal Q-automaton as its behavior, we see that the category δ-Aut(Q) is completely determined by the generic behavior $D(\widetilde{G}\Omega)$. This discussion also shows that describing the category δ-Aut(Q) as a coreflective subcategory of $\text{Aut}(Q)$ and describing the object $D(\widetilde{G}\Omega)$ are under some extent equivalent. The latter approach was the one pursued in an earlier version of this paper [30].

5.2 Extensiveness is Necessary

Finally, the reader should notice that all over section 4 we have implicitly used extensiveness of $\mathcal{C}$: we have assumed that every Q-automaton comes with an associated F-algebra labeled by Ω so that diagrams such as

$$
\begin{array}{ccc}
Q_i^{\mathcal{A}} \xrightarrow{\ h_i\ } \Omega_i & \qquad & Q_j^{\mathcal{A}} \xrightarrow{\ f_j\ } Q_j^{\mathcal{B}} \\
\mathtt{in}_i \Big\downarrow \qquad \Big\downarrow \mathtt{in}_i & & \mathtt{in}_j \Big\downarrow \qquad \Big\downarrow \mathtt{in}_j \\
Q^{\mathcal{A}} \xrightarrow{\ h\ } \Omega & & Q^{\mathcal{A}} \xrightarrow{\ f\ } Q^{\mathcal{B}}
\end{array}
$$

are pullbacks. If extensiveness were not used at this point, we could use the diagram of reflective and coreflective subcategories

$$
\text{CoAlg}(P) \xrightarrow{\ \perp\ } \text{Aut}(P) \xrightarrow{\ \perp\ } \delta\text{-Aut}(Q) \xrightarrow[\perp]{} \text{Aut}(Q) \xleftarrow{\ \perp\ } \text{CoAlg}(Q)
$$

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to argue that the existence of a terminal object in $\mathrm{CoAlg}(Q)$ gives rise to a terminal object in $\mathrm{CoAlg}(P)$. It is not the case, however, that the construction $D(\widetilde{G}\Omega)$ gives rise to a final P-coalgebra if coproducts are not disjoint. In this sense, extensiveness of $\mathcal{C}$ is a necessary condition. For example, if we consider an arbitrary Heyting algebra, we need to ask whether the relation

$$\nu_X.(\bigvee_{i \in I} (\Omega_i \wedge G_i X)) = \Omega_I \wedge \bigwedge_{i \in I} \widetilde{G}(\Omega_i \to G_i \Omega_I) \tag{6}$$

holds, that is, whether the expression on the right – which is the posetal version of the pullback of Ω_I against $D(\widetilde{G}\Omega)$ – produces a greatest fixed point of the function $\bigvee_i \Omega_i \wedge G_i X$. When the indexing set I is a singleton this equation holds.[6] On the other hand, the transition system below provides a counterexample to the binary version:

$$\Omega_1, \Omega_2$$
$$a \xrightarrow{\quad 1 \quad} b$$

Here, the transition system has two states, a and b, and a transition from a to b labeled by 1. Over this transition system we interpret the modal operator $\langle 1 \rangle$, by means of the relation labeled by 1, and a second modal operator $\langle 2 \rangle$ by means of the empty relation. We also declare that the two propositional constants Ω_1 and Ω_2 hold in state b only. If we put $\langle \{1,2\}^* \rangle Y = \mu_X.(Y \vee \langle 1 \rangle X \vee \langle 2 \rangle X)$, then we observe that the relation

$$(\Omega_1 \wedge \Omega_2) \vee \langle \{1,2\}^* \rangle (\neg \Omega_1 \wedge \langle 1 \rangle (\Omega_1 \wedge \Omega_2))$$
$$\vee \langle \{1,2\}^* \rangle (\neg \Omega_2 \wedge \langle 2 \rangle (\Omega_1 \wedge \Omega_2))$$
$$\leq \mu_X.((\Omega_1 \vee \langle 1 \rangle X) \wedge (\Omega_2 \vee \langle 2 \rangle X))$$

does not hold in the transition system: this relation is the dual of (6).

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